MAT 1322 C Winter 2014. Wednesday, March 12 17h30–18h50
Termeh Kousha

Midterm 2, Version A

Last name [CAPITAL]: **First name:**

Student Number : ______________________________

- Length: 80 minutes.
- Only those calculators explicitly allowed by the Faculty of Sciences (Texas Instruments TI-30, TI-34 et Casio fx-260, fx-300) are authorized. This is a closed-book, closed-note test.
- Answer each question in the space provided. Use the back of the page for scratch work if necessary. Questions 1 to 4 are multiple choice, each worth 2 points, place your answers to the multiple choice questions in the boxes below. Questions 5 to 7 are long-answer questions. The exam is graded out of 20.

Multiple Choice Answers:

#1	#2	#3	#4
B	C	B	A

Multiple Choice Questions

1. [2 pts] The sum of the series $\sum_{n=2}^{\infty} \frac{4}{n^2-1}$ is

A: $\frac{3}{2}$ B: 3 C: $\frac{9}{2}$ D: 2 E: $\frac{11}{2}$ F: 1

$$\sum_{n=2}^{\infty} \frac{4}{(n-1)(n+1)} = \sum_{n=2}^{\infty}\left(\frac{2}{n-1} - \frac{2}{n+1}\right) = \left(2-\frac{2}{3}\right) + \left(1-\frac{1}{2}\right) + \left(\frac{2}{3}-\frac{2}{5}\right)$$

$$+ \cdots \left(\frac{2}{n-2} - \frac{2}{n}\right) + \left(\frac{2}{n-1} - \frac{2}{n+1}\right) + \cdots$$

$$S_n = \sum_{i=2}^{n}\left(\frac{2}{i-1} - \frac{2}{i+1}\right) = 2 + 1 + \cdots - \frac{2}{n} - \frac{2}{n+1}$$

$$\lim_{n\to\infty} S_n = 3$$

2. [2 pts] The sum of the series $3 + 2 + \frac{4}{3} + \frac{8}{9} + \cdots$ is

A: $\frac{2}{3}$ B: $\frac{9}{2}$ C: 9 D: 16 E: $\frac{16}{2}$ F: 8

$a_0 = 3$

$ar = 3(r) = 2 \Rightarrow r = 2/3$

$ar^2 = \frac{4}{3} = 3\left(\frac{2}{3}\right)^2 \qquad r < 1$

$\vdots$

$$S = \frac{a}{1-r} = \frac{3}{1-2/3} = \frac{3}{\frac{1}{3}} = 9$$

3. [2 pts] By using alternating series approach, how many terms at least are required to find the sum of the series $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$ to accuracy 0.0005?

A: $n = 4$	B: $n = 6$ ✓	C: $n = 7$
D: $n = 10$	E: $n = 12$	F: $n = 15$

$R_n < b_{n+1} < 0.0005 = 5\times10^{-4}$

$\frac{1}{(n+1)^4} < 5\times10^{-4} \Rightarrow (n+1)^4 > 2000$

$n+1 > 6.68$

$n > 5.68$

$\Rightarrow n=6$

4. [2 pts] A chocolate donut is taken out of the refrigerator, which has a constant temperature of 4°C and left on the counter. The temperature in the kitchen is constant 20°C. After one hour, the temperature of the donut is 10°C. If the temperature of the donut follows Newton's Law of Heating, what is the temperature after being out of refrigerator for two hours?

A: 13.75°C ✓	B: 14.40°C	C: 16.67°C
D: 17.54°C	E: 19.52°C	F: 20.21°C

$T(t)$: temperature of the chocolate donut.

$T(0)=4$. $T(1)=10$

Newton's law $\quad \frac{dT}{dt} = -K(T-20) \quad K>0 \quad \left(\text{or } \frac{dT}{dt} = K(T-20) \quad K<0\right)$

$\frac{dT}{T-20} = -K\,dt \Rightarrow \ln|T-20| = -Kt + C$

$T-20 = Ae^{-Kt} \Rightarrow T = Ae^{-Kt} + 20$

$T(0) = 4 \Rightarrow 4 = A + 20 \quad A = -16$

$T(t) = -16e^{-Kt} + 20$

$T(1) = 10 \Rightarrow 20 - 16e^{-K} = 10 \Rightarrow -16e^{-K} = -10$

$e^{-K} = 5/8 \Rightarrow K = -\ln(5/8) = 0.47$

$T(2) = 20 - 16e^{-0.47(2)} \simeq 13.75°C$

Long Answer Questions

5. [6 pts] Determine if the following series are convergent or divergent. Justify your answer.

(a) $\sum_{n=1}^{\infty} \frac{5-\sin(n)}{n}$

1

$\frac{5-\sin n}{n} \geqslant \frac{4}{n} > \frac{1}{n}$ or

$\sum_{n=1}^{\infty} \frac{4}{n} = 4 \sum_{n=1}^{\infty} \frac{1}{n}$ is Harmonic Series Divergent

By Comparison Test $\sum \frac{5-\sin n}{n}$ is also Divergent

2 points (b) $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

$f(x) = \frac{x^2}{e^{x^3}}$ Positive, decreasing and cont $\forall x \geqslant 1$ $\underline{|0.5 \text{ points}}$

$\underline{|1 \text{ point}}$ $\int_1^{\infty} \frac{x^2}{e^{x^3}} dx = \lim_{d \to \infty} \int_1^d \frac{x^2}{e^{x^3}} dx = \lim_{d \to \infty} \frac{-1}{3} e^{-x^3} \Big]_1^d = \lim_{d \to \infty} \left(\frac{-1}{3} e^{-d} + \frac{1}{3} e^{-1} \right) = \frac{1}{3e}$ Convergent

so By Integral test (0.5 points) $\sum_{n \geq 1}^{\infty} n^2 e^{-n^3}$ is also convergent

(c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3n + 1}{7n+5}$

1

$\lim_{n \to \infty} \frac{3n+1}{7n+5} = 3/7 \neq 0$ and $\lim (-1)^{n+1} \frac{3n+1}{7n+1} \neq 0$ (DNE)

Divergent.

2 points (d) $\sum_{n=0}^{\infty} \frac{2\,n^n}{n!}$

$\lim_{n \to \infty} \left| \frac{\frac{2(n+1)^{n+1}}{(n+1)!}}{\frac{2\,n^n}{n!}} \right| = \lim_{n \to \infty} \left| \frac{\frac{\cancel{(n+1)}(n+1)^n}{\cancel{(n+1)}\,n!}}{\frac{n^n}{n!}} \right| =$

1 point

$= \lim_{n \to \infty} \left| \left(\frac{n+1}{n} \right)^n \right| = e > 1 \leftarrow$ 0.5 point

0.5 point $\rightarrow$ By Ratio test, divergent and interval

6. [2 pts] Find the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x+1)^n}{3^n(n+3)}$.

0.5 point

$$\lim_{n\to\infty} \left| \frac{\frac{(x+1)^{n+1}}{3^{n+1}(n+4)}}{\frac{x^n}{3^n(n+3)}} \right| = \lim_{n\to\infty} |x+1| \left(\frac{1}{3} \frac{n+3}{n+4} \right) = \frac{|x+1|}{3}$$

So the series is conversent : $\frac{|x+1|}{3} < 1 \Rightarrow |x+1| < 3 \Rightarrow \underline{R=3}$

0.5 point

$$-3 < x+1 < 3 \Rightarrow -4 < x < 2$$

check the end points

0.5 point $x = 2$ $\sum_{n=1}^{\infty} \frac{3^n}{3^n(n+3)} = \sum \frac{1}{(n+3)}$ Divergent (Comparing test to harmoinic series $\sum \frac{1}{n}$)

0.5 point $x = -4$ $\sum_{n=1}^{\infty} \frac{(-3)^n}{3^n(n+3)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{3^n}$ conversent By AST

$$\boxed{-4 \leq x < 2}$$

0.5 point

3

7. [4 pts] (a) For all $-1 < x < 1$, find the power series representation for $\ln(4-x)$ and its interval of convergence. (Hint: $\ln(4-x) = -\int \frac{1}{4-x}dx$. Do NOT forget to find the constant of integration.)

0.5

$$\frac{1}{4-x} = \frac{1}{4}\left(\frac{1}{1-\frac{x}{4}}\right) = \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{x}{4}\right)^n = \sum_{n=0}^{\infty}\frac{x^n}{4^{n+1}} \qquad |x|<4$$

1 point

0.5

$$\ln(4-x) = -\int\sum_{n=0}^{\infty}\frac{x^n}{4^{n+1}}dx = -\sum_{n=0}^{\infty}\int\frac{x^n}{4^{n+1}}dx = -\sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)4^{n+1}} + C$$

$|x|<4$

$x=0 \Rightarrow \ln(4)=C$ 0.5 point

1 point

0.5 point

$$\Rightarrow \ln(4-x) = -\sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)4^{n+1}} + \ln 4 \quad |x|<4$$

1 points

$x=4 \Rightarrow -\sum_{n=0}^{\infty}\frac{1}{n+1}$ D.

$x=-4 \Rightarrow -\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(n+1)}$ C By Ast

$-4 \leq x < 4$

1 point

(b) Use part (a) to express $\ln 3$ as the sum of an infinite series.

let $x=1$

$$\ln 3 = -\sum_{n=0}^{\infty}\frac{1}{(n+1)4^{n+1}} + \ln 4 =$$

or

$$= \ln 4 - \left(\frac{1}{4} + \frac{1}{2\cdot 4^2} + \frac{1}{3\cdot 4^3} + \cdots\right)$$

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Midterm 2, Version B

Last name [CAPITAL]: **First name:**

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Multiple Choice Answers:

$\boxed{\text{C}}$	$\boxed{\text{D}}$	$\boxed{\text{C}}$	$\boxed{\text{C}}$
#1	#2	#3	#4

Multiple Choice Questions

1. [2 pts] The sum of the series $\sum_{k=2}^{\infty} \frac{6}{k^2-1}$ is

A: $\frac{3}{2}$ B: 3 C: $\frac{9}{2}$ ✓ D: 2 E: $\frac{11}{2}$ F: 1

$$\sum_{K=2}^{\infty} \frac{6}{K^2-1} = \sum_{K=2}^{\infty} \frac{3}{K-1} - \frac{3}{K+1} = (3-1)+\left(\frac{3}{2}-\frac{3}{4}\right)+\left(1-\frac{3}{5}\right)$$

$$+ \cdots \left(\frac{3}{K-2} - \frac{3}{K}\right)+\left(\frac{3}{K-1}-\frac{3}{K+1}\right)$$

$$+ \cdots$$

$$S_n = \sum_{K=2}^{n}\left(\frac{3}{K-1} - \frac{3}{K+1}\right) = 3 + 3/2 - \frac{3}{K} - \frac{3}{K+1}$$

$$\lim_{n\to\infty} S_n = 3 + 3/2 = 9/2$$

2. [2 pts] The sum of the series $4 + 3 + \frac{9}{4} + \frac{27}{16} + \cdots$ is

A: $\frac{2}{3}$ B: $\frac{9}{2}$ C: 9 D: 16 ✓ E: $\frac{16}{2}$ F: 8

$a = 4$

$ar = 3 = 4r \Rightarrow r = 3/4$

$ar^2 = \frac{9}{4} = 4\left(\frac{3}{4}\right)^2$

$\vdots$

$$S = \frac{4}{1-3/4} = \frac{4}{\frac{1}{4}} = 16$$

3. [2 pts] By using alternating series approach, how many terms at least are required to find the sum of the series $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ to accuracy 0.002?

A:	$n = 4$	B:	$n = 6$	C: ✓	$n = 7$
D:	$n = 10$	E:	$n = 12$	F:	$n = 15$

$R_n < b_{n+1} < 0.002$

$\frac{1}{(n+1)^3} < 0.002 \Rightarrow (n+1)^3 > 500$

$n+1 > 7.93$

$n > 6.93$

$\boxed{n = 7}$

4. [2 pts] A chocolate donut is taken out of the refrigerator, which has a constant temperature of 4°C and left on the counter. The temperature in the kitchen is constant 22°C. After one hour, the temperature of the donut is 10°C. If the temperature of the donut follows Newton's Law of Heating, what is the temperature after being out of refrigerator for three hours?

A:	13.75°C	B:	14.40°C	C: ✓	16.67°C
D:	17.54°C	E:	19.52°C	F:	20.21°C

$\frac{dT}{dt} = -k(T - 22)$

so (check version A for complete solution)

$T(t) = 22 + Ae^{-kt}$

$T(0) = 4 \Rightarrow A = -18$

$T(1) = 10 = 22 - 18e^{-k}$

$\Rightarrow k = 0.405$

$T(t) = 22 - 18e^{-0.405}$

$T(3) = 22 - 18e^{-0.403(3)} \simeq 16.67°C$

Long Answer Questions

5. [6 pts] Determine if the following series are convergent or divergent. Justify your answer.

1 (a) $\sum_{n=1}^{\infty} \frac{3+\cos(n)}{\sqrt{n}}$

$\frac{3+\cos(n)}{\sqrt{n}} > \frac{3}{\sqrt{n}}$ or $> \frac{1}{\sqrt{n}}$

$\sum \frac{3}{\sqrt{n}}$ or $\sum \frac{1}{\sqrt{n}}$ is Divergent (P- Test $P = +\frac{1}{2} < 1$)

so By Comparison test $\sum \frac{3+\cos(n)}{\sqrt{n}}$ also D

(b) $\sum_{n=1}^{\infty} n^3 e^{-n^4}$

2 $f(x) = x^3 e^{-x^4}$ is conti $[1,\infty)$, positive and decreasing

$\int_1^\infty x^3 e^{-x^4}\,dx = \lim_{d\to\infty} \int_1^d x^3 e^{-x^4}\,dx = \lim_{d\to\infty} -\frac{1}{4} e^{-x^4}\Big|_1^d =$

$\lim_{d\to\infty} -\frac{1}{4}\left[e^{-d^4} + \frac{1}{4}e^{-1}\right] = \frac{1}{4e}$ convergent

so By Integral test $\sum n^3 e^{-n^4}$ is also Convergent

(c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4n^2 + 1}{9n^2 + 2}$

1

$\lim_{n\to\infty} \frac{4n^2+1}{9n^2+2} \neq 0$ $\quad$ $\lim (-1)^{n+1} \frac{4n^2+1}{9n^2+2} \neq 0$ DNE

Divergent.

(d) $\sum_{n=0}^{\infty} \frac{3\, n^n}{n!}$

2

$$\lim_{n\to\infty} \left| \frac{\frac{3(n+1)^{n+1}}{(n+1)!}}{\frac{3n^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{\frac{(n+1)(n+1)^n}{(n+1)\,n!}}{\frac{n^n}{n!}} \right|$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = e > 1$$

Divergent By ratio test

and intervals

6. [2 pts] Find the radius and intervals of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x+2)^n}{4^n(n+4)}$.

$$\lim_{n\to\infty} \left| \frac{\frac{(x+2)^{n+1}}{4^{n+1}(n+5)}}{\frac{(x+2)^n}{4^n(n+4)}} \right| = \frac{|x+2|}{4} \lim_{n\to\infty} \frac{n+4}{n+5} = \frac{|x+2|}{4}$$

convergent when $|x+2| < 4$ $\quad \boxed{R=4}$

$$-4 < x+2 < 4$$
$$-6 < x < 2$$

$x = 2 \Rightarrow \sum_{n=1}^{\infty} \frac{4^n}{4^n(n+4)} = \sum_{n=1}^{\infty} \frac{1}{n+4}$ Diverged

$x = -6 \quad \sum_{n=1}^{\infty} \frac{(-4)^n}{4^n(n+4)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+4)}$ Converg By AST

$$\boxed{-6 \leq x < 2}$$

7. [4 pts] (a) For all $-1 < x < 1$, find the power series representation for $\ln(3-x)$. (Hint: $-\ln(3-x) = \int \frac{1}{3-x}dx$. Do NOT forget to find the constant of integration.)

$$\frac{1}{3-x} = \frac{1}{3}\,\frac{1}{1-\frac{x}{3}} = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n = \sum \frac{x^n}{3^{n+1}} \quad \left|\frac{x}{3}\right|<1 \quad |x|<3$$

$$\ln(3-x) = -\int \frac{1}{3-x}\,dx = -\int \sum_{n=0}^{\infty} \frac{x^n}{3^{n+1}}dx$$

$$= -\sum_{n=0}^{\infty}\int \frac{x^n}{3^{n+1}}dx = -\sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)3^{n+1}} + C \qquad |x|<3$$

$$x=0 \Rightarrow \ln 3 = C \Rightarrow \ln(3-x) = -\sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)3^{n+1}} + \ln 3$$

end points

$$x=3 \quad -\sum_{n=1}^{\infty}\frac{3^{n+1}}{(n+1)3^{n+1}} \quad \sum_{n=1}^{\infty}\frac{1}{n+1} \quad D$$

$$x=-3 \quad -\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n+1} = \text{Convergent} \quad \text{By AST}$$

$$\underline{-3 \leq x < 3}$$

(b) Use part (a) to express $\ln 2$ as the sum of an infinite series.

let $x=1$

$$\ln 2 = -\sum_{n=0}^{\infty}\frac{1}{(n+1)3^{n+1}} + \ln 3$$

or

$$\ln 3 - \left(\frac{1}{3} + \frac{1}{2\cdot 3^2} + \frac{1}{3\cdot 3^3} + \cdots\right)$$